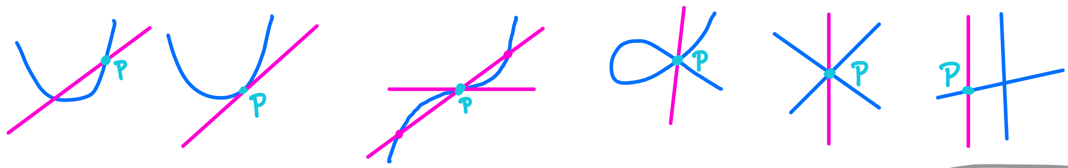
§ 3.3. Intersection Numbers (of two plane curves)



$\approx \# V(F, G)$

F, G = plane curves. $P \in \mathbb{A}^2$

aim: define the Intersection number $I(P, F\cap G)$ of F and G at P.

Seven properties

calculating $I(P, F\cap G)$

F and G **intersect properly** at P if F and G have no common component passes through P.

F and G **intersect transversally** at P if

- P is simple on both F and G
- tangent line to F at P $\neq$ tangent line to G at P.

(1) $I(P, F\cap G) = \begin{cases} \text{nonnegative integer} & F \& G \text{ int. prp.} \\ \infty & \text{otherwise} \end{cases}$

(2) · $I(P, F\cap G) = 0$ iff $P \notin F \cap G$.

· $I(P, F\cap G)$ depends only on the components of F & G that pass through P.

($\Downarrow$ $F = \underset{\neq 0}{\text{const.}}$ or $G = \overset{\neq 0}{\text{const.}}$ $\Rightarrow$ $I(P, F\cap G) = 0$)

(3). T = affine change of coordinates on $\mathbb{A}^2$. then

$$I(P, F\cap G) = I(T^{-1}(P), F^T \cap G^T)$$

(4). $I(P, F\cap G) = I(P, G\cap F)$

(5). $I(P, F\cap G) \geq m_P(F)\cdot m_P(G)$,

"=" $\Leftrightarrow$ (tangent line to F at P $\neq$ tangent line to G at P)

(6). $F = \prod F_i^{r_i}$, $G = \prod G_j^{s_j}$ then

$$I(P, F\cap G) = \sum_{i,j} r_i s_j \; I(P, F_i \cap G_j).$$

(7). F = irr. $\Rightarrow$ $I(P, F\cap G) = I(P, F\cap (G+AF))$ $\forall A \in k[x,y]$.

Thm. $\exists!$ $I(P, F\cap G)$ for all F, G & all $P \in \mathbb{A}^2$ satisfying (1)–(7).

it is given by

$$I(P, F\cap G) = \dim_k \left(\mathcal{O}_P(\mathbb{A}^2) / (F, G) \right)$$

Pf: uniqueness. ONTS: one can calculate $I(P, F\cap G)$ for any F, G, P.

- (3) $\xRightarrow{\text{WMA}}$ $P = (0,0)$ $\xRightarrow[\text{WMA}]{(1)}$ $I(P, F\cap G) < \infty$
- (2) $\Rightarrow$ $(I(P, F\cap G) = 0 \Leftrightarrow P \notin F\cap G)$
- Inductively. assume $I(P, F\cap G) = n$ and $I(P, A\cap B)$ can be calculated whenever $I(P, A\cap B) < n$.

$$r = \deg_x F(x,0), \quad s := \deg_x G(x,0)$$

(4) $\xRightarrow{\text{WMA}}$ $r \leq s$.

1° $r=0$. (and assume $G(X,0)=X^m(a_0+a_1X+\dots)$)

$\Rightarrow Y|F \Rightarrow F=YH$

$$\Rightarrow I(P,F\cap G) \overset{(6)}{=} I(P,Y\cap G)+I(P,H\cap G)$$
$$\overset{(7)}{=} I(P,Y\cap G(X,0))+I(P,H\cap G)$$
$$\overset{(2)}{=} I(P,Y\cap X^m)+I(P,H\cap G)$$
$$\overset{(6)(5)}{=} m+I(P,H\cap G)$$

$\Rightarrow I(P,H\cap G)=I(P,F\cap G)-m<n$

$\Rightarrow I(P,H\cap G)$ can be calculated with (1)-(7)

$\Rightarrow I(P,F\cap G)$ can be calculated with (1)-(7).

2° $r\geq 1$. WMA: $F(X,0)$ & $G(X,0)$ monic. $H:=G-X^{s-r}F$

$$\Rightarrow \begin{cases} I(P,F\cap G) \overset{(7)}{=} I(P,F\cap H) \\ \deg(H(X,0))=:t<s \end{cases}$$

repeating $\Rightarrow$ case 1°.

Existence: $I(P,F\cap G):=\dim_k\left(\mathcal{O}_P(\mathbb{A}^2)/(F,G)\right)$

WANTS: $I(P,F\cap G)$ satisfies (1)-(7).

- $I(P,F\cap G)$ only depends on $(F,G)\trianglelefteq \mathcal{O}_P(\mathbb{A}^2) \Rightarrow$ (2), (4), (7)
- affine change $\Rightarrow$ isomorphism of local ring $\Rightarrow$ (3).

WLOG, WMA: $P=(0,0)$ all components of F & G pass through P.

$\mathcal{O}:=\mathcal{O}_P(\mathbb{A}^2)$

$$\gcd(F,G)=1 \overset{\S1.6}{\Rightarrow} V(F,G)<\infty \overset{\S1.7\text{ cor }4}{\Rightarrow} \dim k[x,y]/(F,G)<\infty \overset{\text{cor }1 \text{ in } \S2.9}{\Rightarrow} I(P,F\cap G)<\infty.$$

$$\left(\begin{array}{l} \text{or if } \underset{\text{const.}}{\underset{\neq}{H}} \mid \gcd(F,G) \Rightarrow (F,G)\subset(H) \Rightarrow I(P,F\cap G) \geq \dim_k(\mathcal{O}/(H)) \\ \qquad\qquad \mathcal{O}/H \cong \mathcal{O}_P(H) \supseteq P(H) \\ \qquad \Rightarrow I(P,F\cap G)=\infty. \Rightarrow (1). \end{array}\right)$$

Proof of (6): ONTS : $I(P,F\cap GH) = I(P,F\cap G) + I(P,F\cap H)$

WMA : $\gcd(F,GH)=1$.

ONTS: $0 \to \mathcal{O}/(F,H) \xrightarrow{\psi} \mathcal{O}/(F,GH) \xrightarrow{\varphi} \mathcal{O}/(F,G) \to 0$ exact.

$\bar{z} \mapsto \overline{Gz}$

$\bar{z} \in \ker\varphi \Leftrightarrow \bar{z} \in \operatorname{Im}\psi$ (easy to check)

ONTS: ψ is injective.

$\forall\, z\in\mathcal{O}$ s.t. $\psi(\bar{z})=0 \Leftrightarrow Gz = Fu + GHv$ for some $u,v\in\mathcal{O}$

$\underset{sz,\,su,\,sv\in k[x,y]}{\overset{\exists s\in k[x,y] \text{ s.t. } s(P)\neq 0 \,\&}{\Longleftrightarrow}} \quad G\cdot(sz) = F\cdot(su) + GH\cdot(sv)$

$\Rightarrow G \mid su \ \&\ z = F\cdot\frac{su}{G}\cdot\frac{1}{s} + H\cdot v \in (F,H)\triangleleft\mathcal{O}$

$\Rightarrow \bar{z}=0$ in $\mathcal{O}/(F,H)$.

Proof of (5): $m := m_P(F)$, $n = m_P(G)$. $I=(x,y)\triangleleft k[x,y]$.

$$\begin{array}{ccccccc} k[x,y]/I^n \times k[x,y]/I^m & \xrightarrow{\psi} & k[x,y]/I^{m+n} & \overset{\varphi}{\twoheadrightarrow} & k[x,y]/(I^{m+n},F,G) & \to 0 & \underline{\text{exact}} \\ (\bar{A},\bar{B}) & \longmapsto & \overline{AF+BG} & & \cong\big\downarrow \alpha \Leftarrow \text{Cor 2 (2.9)} & & \\ & & \mathcal{O}/(F,G) & \overset{\pi}{\twoheadrightarrow} & \mathcal{O}/(I^{m+n},F,G) & \to 0 & \end{array}$$

$$\Rightarrow \quad \dim\left(k[X,Y]/I^n\right) + \dim\left(k[X,Y]/I^m\right) \geq \dim(\ker\varphi)$$

$$\text{"="} \Leftrightarrow \psi = \text{inj}$$

$$\dim\left(k[X,Y]/(I^{m+n}, F, G)\right) = \dim\left(k[X,Y]/I^{m+n}\right) - \dim\ker(\varphi)$$

$$\Rightarrow \quad I(P, F\cap G) = \dim\left(\mathcal{O}/(F,G)\right)$$

$$\geq \dim\left(\mathcal{O}/(I^{m+n}, F, G)\right)$$

$$= \dim\left(k[X,Y]/(I^{m+n}, F, G)\right)$$

$$\geq \dim\left(k[X,Y]/I^{m+n}\right) - \dim\left(k[X,Y]/I^n\right) - \dim\left(k[X,Y]/I^m\right)$$

$$= \frac{(m+n)(m+n+1)}{2} - \frac{n(n+1)}{2} - \frac{m(m+1)}{2}$$

$$= mn.$$

$$\text{"="} \Leftrightarrow \begin{cases} \pi = \text{Iso}. \\ \psi = \text{inj} \end{cases}$$

$\Rightarrow$ (5) follows:

Claim: a) F & G has no common tangents at P then

$$I^t \subset (F,G)\mathcal{O} \text{ for } t \geq m+n-1.$$

b) $\psi = \text{inj} \Leftrightarrow$ F & G has distinct tangents at P.

pf of a): $L_1, \dots, L_m$ tangents to F at P $(L_i = L_m$ for $i > m)$

$M_1, \dots, M_n$ tangents to G at P $(M_j = M_n$ for $j > n)$

$$A_{ij} := L_1 \cdots L_i M_1 \cdots M_j \quad \forall\, i, j \geq 0.$$

Prob 2.35 (c) $\Rightarrow \{A_{ij} \mid i+j = t\}$ = basis for vec. space of all forms of degree t in $k[X,Y]$.

$\Rightarrow$ ONTS: $A_{ij} \in (F,G)\mathcal{O}$ for all $i+j \geq m+n-1$.

$$A_{ij} = \begin{cases} A_{m0} B & i \geq m \\ A_{0n} B & j \geq n \end{cases}$$ (for some form B, of deg $= i+j-m$ or $i+j-n$)

WLOG, WMA: $A_{ij} = A_{m0} B$.

$$F = A_{m0} + F' \; (\text{deg of terms of } F' \geq m+1)$$

$\Rightarrow A_{ij} \equiv -BF' \pmod{F, G}$ deg of terms of $BF' \geq i+j-m+m+1 \geq i+j+1$

ONTS: $I^t \subset (F,G)\mathcal{O}$ for $t \gg 0$.

$V(F,G) = \{P, Q_1, \dots, Q_s\}$

H s.t. $H(P) \neq 0$ & $H(Q_i) = 0$ $\forall i = 1, \dots, s$

$\Rightarrow HX, HY \in I(V(F,G))$

$\Rightarrow (HX)^N, (HY)^N \in (F,G)$ for some N.

$\Rightarrow I^{2N} \subseteq (F,G)$. $\square$

Proof of (b): Suppose tangents are distinct and

$$\psi(\bar{A}, \bar{B}) = \overline{AF + BG} = 0.$$

12 i.e. deg. of terms in $AF + BG \geq m+n$. Assume

$$\begin{cases} A = \overset{\neq 0}{A_r} + \text{higher terms} & (r < n) \\ B = B_s + \text{higher terms} & (s < m) \end{cases}$$

$$AF + BG = A_r F_m + B_s G_n + (\text{higher terms}) \in I^{m+n} \quad \left(\Rightarrow B_s \neq 0.\right)$$

$$\Rightarrow \begin{cases} r+m = s+n \\ A_r F_m = -B_s G_n \end{cases}$$

no common tangent $\Rightarrow \gcd(F_m, G_n) = 1$

$$\Rightarrow \begin{cases} F_m \mid B_s \Rightarrow m \leq s \\ G_n \mid A_r \Rightarrow n \leq r \end{cases} \quad ↯$$

Conversely, assume $L \mid \gcd(F_m, G_n)$ i.e.

$$\begin{cases} F_m = L F'_{m-1} \\ G_n = L G'_{n-1} \end{cases} \quad \text{Then}$$

$$\varphi\left(\overline{G'_{n-1}}, -\overline{F'_{m-1}}\right) = 0 \Rightarrow \varphi \neq \text{inj.}$$

about minimal set of axioms:

① We only need $I((0,0), X \cap Y) = 1$ in (5) !

②. calculation should be easy.

Example: $E = (X^2+Y^2)^2 + 3X^2Y - Y^3$

$F = (X^2+Y^2)^3 - 4X^2Y^2 \qquad P = (0,0)$

解: $F - (X^2+Y^2)E = -4X^2Y^2 - (X^2+Y^2)(3X^2Y - Y^3)$

$$= Y[\underbrace{(X^2+Y^2)(Y^2-3X^2) - 4X^2Y}_{G}]$$

$$G + 3E = Y(\underbrace{5X^2 - 3Y^2 + 4Y^3 + 4X^2Y}_{H})$$

$$\gcd(\underbrace{3X^2Y - Y^3}_{E_3}, \underbrace{5X^2 - 3Y^2}_{H_2}) = 1$$

$$\Rightarrow I(P, E\cap H) = m_P(E)\cdot m_P(H) = 3\cdot 2 = 6$$

$$I(P, E\cap Y) = I(P, X^4 \cap Y) = 4$$

$$\Rightarrow I(P, E\cap F) = 2I(P, E\cap Y) + I(P, E\cap H)$$

$$= 2\times 4 + 6 = 14$$

Lem: (8). P = simple on $F \Rightarrow I(P, F\cap G) = \operatorname{ord}_P^F(G)$

(9). $\gcd(F,G) = 1 \Rightarrow \sum_P I(P, F\cap G) = \dim_k(k[X,Y]/(F,G))$

Pf (8): WMA: F = irr. $g = \bar{G} \in \mathcal{O}_P(F) \Rightarrow \mathcal{O}_P(F)/(g) \overset{\text{Prob 2.44}}{\cong} \mathcal{O}_P(\mathbb{A}^2)/(F,G)$

Prob 2.50(c) $\Rightarrow \operatorname{ord}_P^F(G) = \dim_k(\mathcal{O}_P(F)/(g)) = I(P, F\cap G)$

(9). cor 1. (§2.9).